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AN EXACT SOLUTION TO THE BOUNDARY LAYER EQUATIONS OF INDUCED MOTION FROM A SPREADING POINT SOURCE

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Abstract

This paper considers the axisymmetric spreading of material from a point source on the surface of a higher-dense liquid, similar to oil spreading on the surface of water or the spreading of a light-weight powder. The nature of the induced motion in the boundary layer adjacent to the stretching surface is investigated, including the similarity solution, f(ζ) integrated in a closed form, exact solution for the radial velocity u(r, z), w(r, z→∞) to be evaluated, explicit equation for δ(r), and the frictional force τ of the liquid on the solid is computed as well.

Key words: exact solution, radial velocity, similarity solution, Stokes stream function

I. INTRODUCTION

There have been studies that use approximation of Navier-Stokes equation for viscous motion of fluid (Eggers and Dupont, 1994), numerical solution for Falkner-Skan flow (Elgazery, 2008; Abbasbandy et al., 2014), similarity solution of Falkner-Skan equations for flow driven over a stretching boundary (Riley and Weidman, 1989; Postelnicu and Pop, 2011), similarity solution that reduces the Navier-Stokes equations (Weidman, 2015; Weidman and Perocco, 2016), and exact solution of Navier-Stokes equations (Wang, 2002). Recently, Herrmann-Priesnitz et al. (2016) formulated a flow model of radial inflow. In this paper, the authors wish to determine the simplified similarity solution of boundary-layer equations of radial outflow.

II. PROPOSITION

The volume flow rate of the material Q is assumed to be constant and the material layer flows radially outward on the fluid surface at constant thickness h. Under the specified condition, the authors show that conservation of mass of the spreading material requires its spreading velocity to be inversely proportional to the radius, namely

$$Q = \mathrm{U(r)}\ A = \mathrm{U(r)} \int_0^h \int_0^{2\pi} \mathrm{r}\, d\theta\, dz = \mathrm{U(r)}\ 2\,\pi\,\mathrm{r}\, h$$

$$\mathrm{U(r)} = \frac{Q}{2\pi h}\frac{1}{\mathrm{r}} = \frac{\alpha}{\mathrm{r}} \qquad \text{Eq. (1.1)}$$

Far from the origin, the authors investigate the nature of the induced motion in the boundary layer adjacent to the moving (stretching) surface, presumably governed by the boundary layer equations

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=v\frac{\partial^2 u}{\partial z^2} \text{ (axial momentum)} \qquad \text{Eq. (1.2)}$$

$$\frac{\partial(r\,u)}{\partial r}+\frac{\partial(r\,w)}{\partial z}=0 \text{ (continuity)}$$

where the boundary conditions are

u = U(r) at z = 0; w = 0 at z = 0; and u → 0 at z → ∞. Eq. (1.3)

In this paper, the authors posit the similarity solution in the form of

$$u=U(r)f'(\zeta) \text{ where } \zeta=\frac{z}{\delta(r)}. \qquad \text{Eq. (1.4)}$$

Then, the authors would solve δ(r) which allows for a similarity solution. The authors would show that the equation for f(ζ) may be integrated in closed form and find the exact solution for the radial velocity u(r, z). Evaluation of w(r, z→∞) is to be followed. By defining the boundary layer δ as the value at which the radial velocity is 0.01U(r), the authors would obtain the explicit equation for δ(r). Finally, the authors would compute the frictional force τ of the liquid on the solid.

III. EXISTENCE AND UNIQUENESS OF SIMILARITY SOLUTIONS

The authors introduce a Stokes stream function and find its governing equation

$$u=-\frac{1}{r}\frac{\partial\psi}{\partial z}\;;\quad w=\frac{1}{r}\frac{\partial\psi}{\partial r}$$

$$u_r=\frac{\psi_z}{r^2}-\frac{\psi_{rz}}{r}\;;\quad u_z=-\frac{\psi_{zz}}{r}\;;\quad u_{zz}=-\frac{\psi_{zzz}}{r}$$

Hence,

$$\left(\frac{\psi_z}{r^2}\right)\left[\frac{\psi_z}{r^2}-\frac{\psi_{rz}}{r}\right]+\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\left(-\frac{\psi_{zz}}{r}\right)=v\left(-\frac{\psi_{zzz}}{r}\right)$$

$$\frac{v}{r}\psi_{zzz}-\frac{1}{r^3}\psi_z^2+\frac{1}{r^2}\psi_z\psi_{rz}-\frac{1}{r^2}\psi_r\psi_{zz}=0$$

The governing equation may also be expressed by the form of Ψ(r, z) from Eq. (1.4).

$$u=U_0(r)\,f'(\zeta)=-\frac{1}{r}\frac{\partial\psi}{\partial z}$$

$$\frac{\partial\psi}{\partial z}=-r\,U_0(r)\,f'(\zeta)$$

Since $\zeta_r=-\frac{z\,\delta'}{\delta^2}=-\frac{\zeta\,\delta'}{\delta^2}$, $\frac{\partial f}{\partial z}=f'\,\zeta_r=\frac{f'}{\delta}$, and $f'=\delta\frac{\partial f}{\partial z}$,

Then, $$\psi\approx-\alpha\int\frac{df}{d\zeta}dz$$

$$\psi\approx-\alpha\,\delta(r)\,f(\zeta)$$

$$\therefore\quad \psi(r,z)=-\alpha\,\delta(r)\,f(\zeta)$$

Thus, the condition for similarity is $\psi(r,z)=-\alpha\,\delta(r)\,f(\zeta)$.

Also,

$$\psi_z = -\alpha\,\delta\,f'\frac{1}{\zeta}\ ;\ \ \psi_{zz} = -\alpha\,\delta\,f''\frac{1}{\zeta^2}\ ;\ \ \psi_{zzz} = -\alpha\,\delta\,f'''\frac{1}{\zeta^3}$$

$$\psi_r = -\alpha\,\delta' f - \alpha\,\delta\,f'\left(\frac{\zeta\,\delta'}{\delta}\right) = -\alpha\,\delta'\left(f - \zeta\,f'\right)$$

$$\psi_{rz} = -\alpha\,\delta'\left(f' - f' - \zeta\,f''\right)\zeta_z = \alpha\frac{\delta'}{\delta}\zeta\,f''$$

Then,

$$\frac{\nu}{r}\left(-\frac{\alpha\,f'''}{\delta^2}\right) - \frac{1}{r^3}\left(-\alpha\,f'\right)^2 + \frac{1}{r^2}\left(-\alpha\,f'\right)\left(\frac{\alpha\delta'}{\delta}\zeta\,f''\right) - \frac{1}{r^2}\left[-\alpha\,\zeta\left(f - \zeta\,f'\right)\right]\left(-\frac{\alpha\,f''}{\delta}\right) = 0$$

$$-\frac{\nu\alpha}{r\delta^2}f''' - \frac{\alpha^2}{r^3}\left(f'\right)^2 - \frac{\alpha^2}{r^2}\frac{\delta'}{\delta}\delta\,f'\,f'' - \frac{\alpha^2}{r^2}\frac{\delta'}{\delta}f\,f'' + \frac{\alpha^2}{r^2}\frac{\delta'}{\delta}\delta\,f'\,f'' = 0$$

$$f''' - \left(\frac{\alpha}{\nu}\right)\frac{\delta^2}{r^2}\left(f'\right)^2 + \left(\frac{\alpha}{\nu}\right)\frac{\delta'\delta}{r}ff'' = 0$$

By setting the condition for continuity that $\frac{\delta^2}{r^2} = \text{const}$ and $\frac{\delta'\delta}{r} = \text{const}$, to obtain $\delta(r) = r$.

With the boundary conditions of Eq. (1.3),

$$w = \frac{1}{r}\psi_r = -\frac{\alpha\,\delta'}{r}\left(f - \zeta\,f'\right) = -\frac{\alpha}{r}\left(f - \zeta\,f'\right).$$

$f(0) = 0$ and $f'(\infty) \to 0$ from the conditions that u → 0 and z → ∞.

Thus, to solve

$$\frac{\nu}{\alpha}f''' + \left(f'\right)^2 + f\,f'' = 0$$

where $\delta = \frac{z}{r}$, $f(0) = 0$, $f'(0) = 1$, and $f'(\infty) \to 0$.

By integration, $$\frac{\nu}{\alpha}f''' + \left(f\,f'\right)' = 0.$$

$$\frac{\nu}{\alpha}f'' + f\,f' = C_1$$

With the boundary condition $C_1 = 0$,

$$\frac{\nu}{\alpha}f'' + \left(\frac{f^2}{2}\right) = 0$$

$$\frac{\nu}{\alpha}f' + \frac{f^2}{2} = C_2$$

With the boundary conditions that $f(0) = 0$ and $f'(0) = 1$,

$$C_2 = \frac{\nu}{\alpha}.$$

Then,

$$\frac{\nu}{\alpha}\frac{df}{d\zeta} + \frac{f^2}{2} = \frac{\nu}{\alpha}$$

$$\frac{\nu}{\alpha}\frac{d\mathrm{f}}{d\zeta}=\frac{\nu}{\alpha}-\frac{\mathrm{f}^2}{2}$$

Continuing

$$\int\frac{d\mathrm{f}}{\frac{\nu}{\alpha}-\frac{\mathrm{f}^2}{2}}=\frac{\nu}{\alpha}\int d\zeta+C_3=2\int\frac{d\mathrm{f}}{\left(\frac{2\nu}{\alpha}-\mathrm{f}^2\right)}$$

$$\frac{1}{\sqrt{\frac{2\nu}{\alpha}}}\tanh^{-1}\left(\frac{\mathrm{f}}{\sqrt{\frac{2\nu}{\alpha}}}\right)=\frac{\alpha}{2\nu}\zeta+C_3'$$

From f(0) = 0, $C_3'=0$.

$$\tanh^{-1}\left(\frac{\mathrm{f}}{\sqrt{\frac{2\nu}{\alpha}}}\right)=\frac{\alpha}{2\nu}\sqrt{\frac{2\nu}{\alpha}}\zeta=\sqrt{\frac{\alpha}{2\nu}}\zeta$$

$$\mathrm{f}=\sqrt{\frac{2\nu}{\alpha}}\tanh\left\{\sqrt{\frac{\alpha}{2\nu}}\zeta\right\}$$

Thus,

$$\mathrm{U(r,z)}=\frac{\alpha}{\mathrm{r}}\mathrm{f}'(\zeta)$$

$$\mathrm{f}'(\zeta)=\sqrt{\frac{2\nu}{\alpha}}\mathrm{sech}^2\left(\sqrt{\frac{\alpha}{2\nu}}\zeta\right)\sqrt{\frac{\alpha}{2\nu}}=\mathrm{sech}^2\left(\sqrt{\frac{\alpha}{2\nu}}\zeta\right)$$

Thus,

$$\mathrm{U(r,z)}=\frac{\alpha}{\mathrm{r}}\mathrm{sech}^2\left(\sqrt{\frac{\alpha}{2\nu}}\zeta\right)\quad\text{where}\quad\zeta=\frac{\mathrm{z}}{\mathrm{r}}.$$

To determine the velocity at an infinite (∞),

$$\mathrm{w(r,\infty)}=-\frac{\alpha}{\mathrm{r}}\left[\mathrm{f}-\zeta\,\mathrm{f}'\right]_{\zeta\to\infty}$$

Note that

$$\mathrm{f}'(\zeta)=\mathrm{sech}^2\left(\sqrt{\frac{\alpha}{2\nu}}\zeta\right)=\left(\frac{2}{\exp\left(\sqrt{\frac{\alpha}{2\nu}}\zeta\right)+\exp\left(-\sqrt{\frac{\alpha}{2\nu}}\zeta\right)}\right)^2$$

Thus,

$$\mathrm{f}'(\zeta)\approx 4\exp\left(-\frac{\alpha}{2\nu}\zeta^2\right)\quad\text{for}\quad\zeta\to\infty.$$

And,

$$\lim_{\zeta\to\infty}\left[\zeta\,\mathrm{f}'(\zeta)\right]=\lim_{\zeta\to\infty}\left[4\zeta\exp\left(-\frac{\alpha}{2\nu}\zeta^2\right)\right]=0$$

Consequently,

$$w(r,\infty) = -\frac{\sqrt{2\nu\alpha}}{r} \quad \text{where } \alpha = \frac{Q}{2\pi h}$$

The upward velocity outside boundary layer necessary to provide mass flux inside the linearly growing boundary layer is illustrated in the sketch of Figure 2.1.

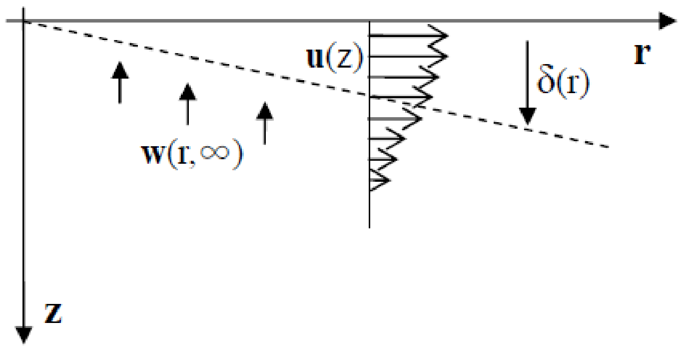

Figure 2.1

When the boundary layer thickness is defined as the point in the z-direction where $U = 0.01\,U_0(r)$, hence having

$$\frac{U(r,z)}{U_0(r)} = \text{sech}^2\left(\sqrt{\frac{\alpha}{2\nu}}\,\zeta\right) = 0.01 \quad \text{where } \zeta = \frac{\delta}{r}.$$

Then it is obtained that $\quad \text{sech}^2\left(\sqrt{\frac{\alpha}{2\nu}}\frac{\delta}{r}\right) = 0.01$

Let $\quad \beta = \sqrt{\frac{\alpha}{2\nu}}\frac{\delta}{r}$

Solution for this equation gives

$$\text{sech}^2\beta = \frac{1}{\cosh^2\beta} = \left(\frac{2}{\exp(\beta)+\exp(-\beta)}\right)^2 = 0.01$$

$$\frac{2}{\exp(\beta)+\exp(-\beta)} = \sqrt{0.01} = 0.1$$

It is obtained that

$$\exp(\beta)+\exp(-\beta) = \frac{2}{0.1} = 20$$

Let $r = \exp(\beta)$

$$r + \frac{1}{r} = 20$$

Then

$$r^2 - 20\,r + 1 = 0$$

$$r = \frac{20 \pm \sqrt{400-4}}{2} = 10 \pm 9.94987$$

Only solution for $\beta > 0$, it gives r = 19.94987.

Hence, β = Ln(19.94987) = 2.9932.

Recall that $\beta = \sqrt{\frac{\alpha}{2\nu}}\frac{\delta}{r}$

$$\sqrt{\frac{\alpha}{2\nu}}\frac{\delta}{r} = 2.9932$$

Therefore,

$$\delta(r) \cong 3\sqrt{\frac{2\nu}{\alpha}}\, r\,.$$

Derivation for shear stress follows that,

$$\tau = \mu\frac{\partial u}{\partial z}\bigg|_{z=0} = \left[\mu\frac{\partial}{\partial z}\left(\frac{\alpha}{r}\operatorname{sech}\sqrt{\frac{\alpha}{2\nu}}\frac{z}{r}\right)\right]_{z=0}$$

$$\tau \approx 2\operatorname{sech}\left(\sqrt{\frac{\alpha}{2\nu}}\frac{z}{r}\right)\tanh\left(\sqrt{\frac{\alpha}{2\nu}}\frac{z}{r}\right)\bigg|_{z=0} = 0$$

Hence, $C_f = 0$.

IV. CONCLUSIONS

In this paper, the authors have investigated the general solution to the induced motion in the boundary layer far from the origin of the outward radial flow, as well as the upward velocity outside the boundary layer necessary to provide mass flux inside the linearly growing boundary layer. The authors hope that the uniqueness of the general solution is much meaningful for further research of induced viscous flow in mathematical physics.

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